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# EQUATIONS OF HIGHER ORDER OF ACCURACY DESCRIBING THE VIBRATIONS OF THIN PLATES†

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Methods developed in the mathematical theory of the averaging of processes in periodic media are used to derive two-dimensional equations describing the propagation of waves in non-uniform anisotropic plates with a periodic structure. Equations of higher order of accuracy in a small parameter – the ratio of the typical thickness of the plate to the typical wavelength, are derived. The case of uniform isotropic thin plates is considered in detail. Equations of different order of accuracy, derived in this paper, are analysed and compared with the equations proposed by others. Some corrections for the coefficients in Timoshenko-type equations, which increase the accuracy of these equations, are proposed. © 2005 Elsevier Ltd. All rights reserved.

The purpose of this paper is to obtain two-dimensional equations of higher order of accuracy, describing the flexural vibrations of plates with a periodic structure, in particular, laminated anisotropic plates, without any a priori assumptions on the structure of the displacements and deformations in the plate. The solution of this problem enables the numerical solution of three-dimensional non-stationary problems to be validly reduced to the numerical solution of two-dimensional or one-dimensional non-stationary problems, which considerably diminishes the volume of computation required.

Approximate two-dimensional equations of different degrees of accuracy have been derived by many authors for thin plates (see the review devoted to the refined equations of the vibrations of rods and plates in [1]). Existing publications conventionally can be divided into two group. In the first group, certain a priori assumptions are made regarding the plate deformation process. Using the simplest assumptions, the widely known Kirchhoff classical equations are obtain [2], and, taking additional possible effects into account, the so-called exact (non-classical) equations are obtained, in particular, Timoshenko-type equations [1, 3, 4]. The second group is based on the assumption that the displacements can be represented in the form of series (or, with a certain accuracy, by polynomials) in powers of the coordinate perpendicular to the middle plane of the plate. This approach was used, in particular, by Selezov [5] to derive equations of higher order of accuracy for the flexural vibrations of plane uniform isotropic plates.

Below we derive two-dimensional equations of higher order of accuracy using the two-scale expansions method in the form used in the theory of averaging of processes in periodic media [6–9]. In this approach no a priori assumptions are made regarding the nature of the deformation or regarding the form of the dependence of the displacements on the normal coordinate. Instead of this, we use the assumption that the displacements can be represented in the form of asymptotic series in powers of a small parameter $\varepsilon$, equal to the ratio of the typical thickness of the plate to the typical wavelength. When additional limitations are imposed on the data of the problem this assumption can be rigorously justified (see Section 11).

In this way we can obtain two-dimensional equations of any accuracy in $\varepsilon$, and also the distribution of the displacements inside the plate. In the case of the a plane uniform plate, the displacements, in fact, can be represented with any accuracy in the form of a polynomial in powers of the normal coordinate. However, in the case of non-uniform plates the displacement vector may depend on the spatial variables in a more complex way. We used the proposed approach in the general case of plates that are non-uniform in thickness and periodic in longitudinal directions.

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## 1. FORMULATION OF THE THREE-DIMENSIONAL PROBLEM OF THE VIBRATIONS OF THE ELASTIC PLATE

We will call an infinite periodic plate a certain connected set $\Pi$ of a space $R_3$ with a Lipschitz boundary $\partial\Pi$, lying in the strip $|x_3| \leq H_3/2$ and periodic in the variables $x_1$ and $x_2$ with periods $H_1$ and $H_2$ respectively. When simply connectedness requirements is dropped, there is the possibility that there are pores in the plate. We will also assume that the density $\rho(x_1, x_2, x_3)$ and the matrices of the elastic moduli $A_{ij}(x_1, x_2, x_3)$. are measurable, bounded, and periodic in the variables $x_1$ and $x_2$ with periods $H_1$ and $H_2$ respectively. We will further assume that all the quantities $H_j$ are of the same order.

The three-dimensional system of equations of the vibrations of the plate $\Pi$ has the form

$$Lu = -\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} + \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial \mathbf{u}}{\partial x_j}\right) = -\mathbf{F} \tag{1.1}$$

When $\mathbf{F}$ is the density of the bulk forces, and summation is carried out over repeated subscripts in the limits from 1 to 3.

On the surface of the plate, the distribution of the force is specified.

$$A_{ij}\frac{\partial \mathbf{u}}{\partial x_j}n_i\bigg|_{\partial\Pi} = \mathbf{g} \tag{1.2}$$

Here $n_i$ are the components of the vector $\mathbf{n}$ of the normal to the plate surface.

Suppose $\hat{\rho}$ and $\hat{c}$ are the mean values of the density $\rho$ and of the propagation velocity of waves in the components comprising the medium, $\hat{U}$ is a typical value of the displacement and $L$ is the typical length of the wave propagating in the plate. Further constructions are carried out on the assumption that

$$\varepsilon = \frac{H_3}{L} \ll 1$$

Introducing dimensionless variables using the relations

$$t' = \frac{t\hat{c}}{L}, \quad x_j' = \frac{x_j H_3}{L H_j}, \quad \mathbf{u}' = \frac{\mathbf{u}}{\hat{U}} \tag{1.3}$$

we obtain a system of equations and boundary conditions, which differ from system (1.1) with conditions (1.2) in that the quantities without primes are replaced by the corresponding quantities with primes, defined by formulae (1.3) and following expressions

$$A_{ij}' = \frac{A_{ij}\varepsilon^2 L^2}{\hat{\rho} H_i H_j \hat{c}^2}, \quad \rho' = \frac{\rho}{\hat{\rho}}, \quad \mathbf{F}' = \frac{\mathbf{F}L^2}{\hat{\rho}\hat{U}\hat{c}^2}, \quad \mathbf{g}' = \frac{\mathbf{g}L^2\varepsilon}{\hat{\rho}\hat{U}\hat{c}^2\sqrt{n_k^2 H_k^2}}, \quad n_j' = \frac{n_j H_j}{\sqrt{n_k^2 H_k^2}} \tag{1.4}$$

Henceforth we will omit the prime and we will use relations (1.1) and (1.2) for references to the formulation of the problem. After replacing the variables, the functions $\rho$ and $A_{ij}$ will be periodic in the new variables $x_1, x_2$ with periodic $\varepsilon$, and the plate is situated in the strip $|x_3| \leq \varepsilon/2$.

If the functions $\rho, A_{ij}$ and $\mathbf{F}$ do not possess sufficient smoothness, we will understand the satisfaction of relations (1.1) and (1.2) in the sense of Sobolev, namely, we have in mind that $\mathbf{u}$ is a vector function of $H^1_{\text{loc}}$, which satisfies the integral identity

$$\int_{-\infty}^{\infty}\int_{\Pi}\left(-\left(\rho\frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \varphi}{\partial t}\right) + \left(A_{ij}\frac{\partial \mathbf{u}}{\partial x_j}, \frac{\partial \varphi}{\partial x_i}\right) - (\mathbf{F}, \varphi)\right)dx\,dt = \int_{\partial\Pi}(\mathbf{g}, \varphi)ds \tag{1.5}$$

for any finite infinitely differentiable vector function $\varphi(t, x)$.

The purpose of our further constructions will be to transfer from the three-dimensional problem (1.1), (1.2) to two-dimensional equations of the vibrations of plates of high order of accuracy in the dimensionless parameter $\varepsilon$.

## 2. APPLICATION OF THE PROCEDURE OF AVERAGING PROCESSES IN PERIODIC MEDIA TO DERIVE THE TWO-DIMENSIONAL EQUATIONS OF THE VIBRATIONS OF PLATES

To obtain approximate two-dimensional equations of the vibrations of a plate of higher order of accuracy we will use the formal procedure of the method of averaging processes in periodic media [6–8] with the explicit use of the condition that there are no forces applied to the plate surface [9, 10]

$$A_{ij}\frac{\partial \mathbf{u}}{\partial x_j}n_i\Big|_{\partial\Pi} = 0 \tag{2.1}$$

To derive the equations of infinite order of accuracy in $\varepsilon$ in the case of boundary conditions (2.1), we will seek an asymptotic expansion for the displacement in the form

$$\mathbf{u} \sim \mathbf{v} + \sum_{0<q+l_1+l_2}{}' \varepsilon^{q+l_1+l_2} N^q_{l_1l_2}(y_1, y_2, y_3)\big|_{y_j = x_j/\varepsilon}\frac{\partial^{q+l_1+l_2}\mathbf{v}}{\partial t^q \partial x_1^{l_1}\partial x_2^{l_2}} \tag{2.2}$$

where $\mathbf{v} = \mathbf{v}(t, x_1, x_2)$ is a smooth function of the slow variables $t$, $x_1$ and $x_2$ with characteristic scale of variation of the order of unity, $N^q_{l_1l_2}(y_1, y_2, y_3)$ is a $3 \times 3$ matrix, periodic with a periodic of unity, with respect to the fast variables $y_1$ and $y_2$, and $y_j = x_j/\varepsilon$. The characteristic scale of variation of $N^q_{l_1l_2}$ in the variable $x_j$ is of the order of $\varepsilon$. The prime in the notation of the sum here and below denotes that all the integer non-negative $q$, $l_1$ and $l_2$, belonging to the limits under the summation sign, participate in the summation; if these limits are not indicated, the summation is carried out over all the integer non-negative indices $q$, $l_1$ and $l_2$.

The sign $\sim$ in relation (2.2) denotes that the right- and left-hand sides differ by a quantity $O(\varepsilon^n)$ for any value $n$. Below, this sign will sometimes denote closeness of the order of $\varepsilon^n$ for a specific value of $n$, this will be clear from the context. Further, $N^0_{00} = E$ is the identity matrix (or the identity operator), and $N^q_{l_1l_2} = 0$ if at least one of the indices is negative. After substituting series (2.2) into system (1.1) in the case of smooth $\rho$, $A_{ij}$ and $N^q_{l_1l_2}$ we have

$$L\mathbf{u} \sim \sum{}' \varepsilon^{q+l_1+l_2-2} H^q_{l_1l_2}\frac{\partial^{q+l_1+l_2}\mathbf{v}}{\partial t^q \partial x_1^{l_1}\partial x_2^{l_2}} \sim -\mathbf{F}$$

where

$$H^q_{l_1l_2} = L_{yy}N^q_{l_1l_2} + T^q_{l_1l_2}, \quad L_{yy}N = \frac{\partial}{\partial y_i}\left(A_{ij}\frac{\partial N}{\partial y_j}\right)$$

$$T^q_{l_1l_2} = \delta(j)\frac{\partial}{\partial y_i}(A_{ij}N^q_{l_1-\delta_{j1}, l_2-\delta_{j2}}) + \delta(i)A_{ij}\frac{\partial}{\partial y_j}N^q_{l_1-\delta_{i1}, l_2-\delta_{i2}} + \tag{2.3}$$

$$+\delta(i)\delta(j)A_{ij}N^q_{l_1-\delta_{i1}-\delta_{j1}, l_2-\delta_{i2}-\delta_{j2}} - \rho N^{q-2}_{l_1l_2}, \quad \delta(i) = 1-\delta_{i3}$$

After substituting series (2.2) into condition (2.1) we have

$$A_{ij}\frac{\partial \mathbf{u}}{\partial x_j}n_i\Big|_{\partial\Pi} \sim \sum{}' \varepsilon^{q+l_1+l_2-2}\hat{H}^q_{l_1l_2}\frac{\partial^{q+l_1+l_2}\mathbf{v}}{\partial t^q \partial x_1^{l_1}\partial x_2^{l_2}} \sim 0$$

$$\hat{H}^q_{l_1l_2} = \left(A_{ij}\frac{\partial N^q_{l_1l_2}}{\partial y_j} + \delta(j)A_{ij}N^q_{l_1-\delta_{j1}, l_2-\delta_{j2}}\right)n_i\Big|_{\partial\Pi}$$

It follows from relations (2.3) that the coefficient $H^0_{00}$ of $\varepsilon^{-2}$ is equal to zero. We will obtain $N^q_{l_1l_2}$ such that the coefficients $H^0_{10}$ and $H^0_{01}$ for terms of the order of $\varepsilon^{-1}$ will be zero, each of the expressions $H^q_{l_1l_2}$ being equal to a certain constant matrix $h^q_{l_1l_2}$, i.e.

$$H^q_{l_1l_2} = h^q_{l_1l_2} = \text{const}\ \forall q, l_1, l_2 \tag{2.4}$$

and moreover, the following inequalities are satisfied

$$\hat{H}^q_{l_1 l_2} = 0 \ \forall q, l_1, l_2 \tag{2.5}$$

We will denote by $\Omega$ the set of points $y = (y_1, y_2, y_3)$, which satisfy the conditions $(y_1\varepsilon, y_2\varepsilon, y_3\varepsilon) \in \Pi$, and by $\Omega_y$ a periodicity cell, i.e. a set of points $y$ which satisfy the conditions

$$0 < y_1 \le 1, \quad 0 < y_2 \le 1, \quad y \in \Omega$$

Instead of the satisfaction of equalities (2.4) and (2.5) we can require the satisfaction of the integral identity

$$\int\limits_{\Omega_y} \Bigg( -\left( A_{ij} \frac{\partial [N^q_{l_1 l_2}]_m}{\partial y_j} + \delta(i) A_{ij} [N^q_{l_1 - \delta_{j1}, l_2 - \delta_{j2}}]_m, \frac{\partial \varphi}{\partial y_i} \right) - \\ - \left( \delta(j) A_{ij} \frac{\partial}{\partial y_j} [N^q_{l_1 - \delta_{i1}, l_2 - \delta_{i2}}]_m + \delta(i)\delta(j) A_{ij} [N^q_{l_1 - \delta_{i1} - \delta_{j1}, l_2 - \delta_{i2} - \delta_{j2}}]_m - \right. \\ \left. - \rho [N^{q-2}_{l_1 l_2}]_m, \varphi \right) \Bigg) dy = \int\limits_{\Omega_y} ([h^q_{l_1 l_2}]_m, \varphi) \Bigg) dy \ \forall m \tag{2.6}$$

for an arbitrary vector function $\varphi(y) \in H^1(\Omega)$, periodic with a period of unity in the variable $y_1$ and $y_2$. Here $(f, g) = f_1 g_1 + f_2 g_2 + f_3 g_3$ is the usual scalar product and $[A]_m$ is the notation of the $m$th column of the $3 \times 3$ matrix $A$. In the case of non-smooth, in particular, discontinuous $\rho$ and $A_{ij}$, the use of integral identity (2.6) is connected with the essence of the problem, since then the solution $\mathbf{u}$ is determined in a generalized sense, which satisfies identity (1.5).

If relations (2.4) and (2.5) are satisfied, we obtain a system of equations of infinite order with constant coefficients

$$\tilde{L}\mathbf{v} \sim \sum_{q + l_1 + l_2 \ge 2}{}' \varepsilon^{q + l_1 + l_2 - 2} h^q_{l_1 l_2} \frac{\partial^{q + l_1 + l_2} \mathbf{v}}{\partial^q t \partial x_1^{l_1} \partial x_2^{l_2}} \sim -\mathbf{F} \tag{2.7}$$

which can be used instead of system (1.1) with condition (2.1). We have the following in mind: if $\mathbf{v}$ is the asymptotic solution of system (2.7), the displacement $\mathbf{u}$, defined by relation (2.2), asymptotically satisfies relations (1.1) and (2.1). All the above make sense for infinitely differentiable $\mathbf{v}(t, x_1, x_2)$ and $\mathbf{F} = \mathbf{F}(t, x_1, x_2)$. The case when $\mathbf{F}$ also depends on $y_j = x_j/\varepsilon$, is considered in Section 10. In Section 11, we justify the equations of finite order of accuracy in $\varepsilon$ on the assumption of the finite smoothness of $\mathbf{v}$ and $\mathbf{F}$ with respect to $t$, $x_1$ and $x_2$.

Henceforth, when $Q = Q(t, x_1, x_2, x_3, y_1, y_2, y_3)$ we will use the notation

$$\langle Q \rangle = \int\limits_{\Omega_y} Q(t, x_1, x_2, x_3, y_1, y_2, y_3) dy_1 dy_2 dy_3$$

where $t$, $x_1$, $x_2$ and $x_3$ are fixed. Integrating Eq. (2.4) over a period, taking relations (2.3) and (2.5) into account, we obtain

$$\mu h^q_{l_1 l_2} = \langle T^q_{l_1 l_2} \rangle = \\ = \left\langle \delta(i) A_{ij} \left( \frac{\partial}{\partial y_j} N^q_{l_1 - \delta_{i1}, l_2 - \delta_{i2}} + \delta(j) N^q_{l_1 - \delta_{i1} - \delta_{j1}, l_2 - \delta_{i2} - \delta_{j2}} \right) - \rho N^{q-2}_{l_1 l_2} \right\rangle \tag{2.8}$$

where $\mu$ is the measure of the set $\Omega_y$. Note that for the usual assumptions in the theory of elasticity, the satisfaction of the equality $\mu h^q_{l_1 l_2} = T^q_{l_1 l_2}$ also turns out to be sufficient for system (2.4), (2.5) to be solvable in $N^q_{l_1 l_2}$.

We can construct $N^q_{l_1l_2}$ successively, by first ordering the set of indices $(q, l_1, l_2)$, satisfying the order of increase of $q + l_1 + l_2$. The matrices $N^q_{l_1l_2}$ are found from Eqs (2.4) and (2.5) or non-uniquely from integral identity (2.6), apart from a term equal to a certain constant matrix,. We will choose this term so that the equality $\langle N^q_{l_1l_2}\rangle = 0$ is satisfied every time. It can then be seen from the structure of relations (2.3) and (2.8) that $h^q_{l_1l_2} = 0$ when $q$ is odd. Moreover, as follows from the constructions in [8], the matrices $h^q_{l_1l_2}$ satisfy the relation

$$(h^q_{l_1l_2})^T = (-1)^{(l_1+l_2)}h^q_{l_1l_2}$$

If the plane plate is only non-uniform in thickness, i.e. the properties of the material depend only on one spatial variable $y_3 = x_3/\varepsilon$: $\rho = \rho(y_3)$, $A_{ij} = A_{ij}(y_3)$, the matrices $N^q_{l_1l_2}$ depend only on $y_3$. In this case all the $N^q_{l_1l_2}$ and $h^q_{l_1l_2}$ are calculated in quadratures. A special case of this plate is a plate consisting of plane uniform and, in general, anisotropic layers. For such plates the coefficients $h^q_{l_1l_2}$ in Eqs (2.7) were investigated in [11, 12], in which only terms with derivatives no higher than the fourth order were retained. Terms of derivatives of the third and fourth orders are the principal terms responsible for the dispersion of the waves. Types of dispersion of waves in plates with a different number of layers, possessing different types of anisotropy, have been investigated.

## 3. THE EQUATIONS OF THE VIBRATIONS OF A PLANE UNIFORM ISOTROPIC PLATE

We will further consider a plane uniform isotropic plate, occupying the volume $|x_3| \le H_3/2$. We have the following formulae for the dimensional components $(A_{ij})_{kl}$ of the matrices $A_{ij}$

$$(A_{ij})_{kl} = \lambda\delta_{ik}\delta_{jl} + \mu(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$$

where $\lambda$ and $\mu$ are the moduli of elasticity. We will put

$$\hat{\rho} = \rho, \quad \hat{c} = \sqrt{\mu/\rho} \tag{3.1}$$

Since the properties of a plane uniform plate are periodic with any period, we can take $H_1 = H_2 = H_3 = H$. Formulae (1.4) for the dimensionless quantities in this case take the form

$$A'_{ij} = \frac{A_{ij}}{\mu}, \quad \rho' = 1, \quad \mathbf{F}' = \frac{\mathbf{F}L^2}{\mu\hat{U}}, \quad \mathbf{g}' = \frac{\mathbf{g}L}{\mu\hat{U}}, \quad n'_j = n_j$$
$$(A'_{ij})_{kl} = \frac{2\nu}{1-2\nu}\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} \tag{3.2}$$

($\nu$ is Poisson's ratio).

We will investigate flexural vibrations, which are described by the third component $\upsilon_3$ of the vector $\mathbf{v}$. Henceforth we will use the following notation: $D_t$ is the operator of differentiation with respect to $t$, and $\Delta$ is the Laplace operator in the variables $x_1$ and $x_2$. Calculations showed that system (2.7) can be split into a system of equations in the unknowns $\upsilon_1$ and $\upsilon_2$ and an equation in $\upsilon_3$, having the form

$$D_t^2\upsilon_3 = \bar{L}\upsilon_3 + F_3 + O(\varepsilon^8); \quad \bar{L}\upsilon_3 = \sum_{2\le q+l\le 4}{}' \varepsilon^{2q+2l-2}A^{2q}_{2l}D_t^{2q}\Delta^l\upsilon_3 \tag{3.3}$$

where

$$A_0^{2q} = 0 \text{ when } q>1, \quad A_2^2 = \frac{1}{12}, \quad A_4^0 = -\frac{1}{6(1-\nu)}$$
$$A_2^4 = -\frac{1}{120}, \quad A_4^2 = \frac{\nu^2-26\nu+24}{720(1-\nu)^2}, \quad A_6^0 = \frac{\nu-6}{180(1-\nu)^2} \tag{3.4}$$
$$A_2^6 = \frac{17}{20160}, \quad A_4^4 = \frac{18\nu^3-360\nu^2+646\nu-305}{60480(1-\nu)^3}$$

$$A_6^2 = \frac{-\nu^3 + 57\nu^2 - 374\nu + 304}{30240(1-\nu)^3}, \quad A_8^0 = \frac{-3\nu^2 + 34\nu - 101}{15120(1-\nu)^3}$$

In the case of purely flexural vibrations, when $\upsilon_1 = \upsilon_2 = 0$, the third component $u_3$ of the displacement vector $\mathbf{u}$ is related to $\upsilon_3$ as follows:

$$u_3 \sim \upsilon_3 + S\upsilon_3; \quad S\upsilon_3 \sim \sum_{1 \le q+l}' \varepsilon^{2(q+l)} n_{2l}^{2q}\left(\frac{x_3}{\varepsilon}\right) D_t^{2q} \Delta^l \upsilon_3 \tag{3.5}$$

where

$$n_0^2(y) = 0, \quad n_2^0(y) = \frac{\nu(12y^2 - 1)}{24(1-\nu)}, \quad n_0^4(y) = 0$$

$$n_2^2(y) = \frac{240(1-2\nu^2)y^4 + 120(6\nu^2 - 4\nu - 1)y^2 - 54\nu^2 + 40\nu + 7}{11520(1-\nu)^2}$$

$$n_4^0(y) = \frac{240(\nu^2 - 1)y^4 - 120(\nu^2 - 4\nu - 1)y^2 + 7\nu^2 - 40\nu - 7}{5760(1-\nu)^2}$$

Further, when $Q = Q(t, x_1, x_2, x_3)$ we will use the following notation for the mean value of $Q$ over the plate thickness

$$\{Q\} = \frac{1}{\varepsilon} \int_{-\varepsilon/2}^{\varepsilon/2} Q(t, x_1, x_2, x_3) dx_3$$

If $Q = Q_1(x_3/\varepsilon)Q_2(t, x_1, x_2)$, we have $\{Q\} = \langle Q_1 \rangle Q_2$. In relation (3.5) the factor $\dfrac{\partial^{q+l_1+l_2}\upsilon_3}{\partial t^q \partial x_1^{l_1} \partial x_2^{l_2}}$ is independent of the variables $y_3$ and $x_3$, while the factors $n_{2l}^{2q}(x_3/\varepsilon)$ are independent of the variables $x_1$ and $x_2$. Moreover, $\{n_{2l}^{2q}\} = \langle n_{2l}^{2q} \rangle = 0$ as a consequent of the fact that $\langle N_{l_1 l_2}^q \rangle = 0$, when $q + l_1 + l_2 > 0$. Hence

$$\{S\upsilon_3\} \sim 0, \quad V \equiv \{u_3\} \sim \upsilon_3 \tag{3.6}$$

Here the sign $\sim$ denotes equality with accuracy $O(\varepsilon^n)$ for any $n$, and we denote by $V$ the value of the displacement $u_3$ averaged over the thickness.

Note that relation (3.6) holds for any plane plate, both isotropic and anisotropic, that is only non-uniform over the thickness.

## 4. REDUCTION OF THE PROBLEM OF THE VARIATIONS OF A PLATE WITH NON-HOMOGENEOUS BOUNDARY CONDITIONS TO A PROBLEM WITH HOMOGENEOUS BOUNDARY CONDITIONS

We described above a procedure for obtaining two-dimensional equations of the vibrations of a plate in the case when the plate surface is load-free; we will now consider the general case, when $\mathbf{g}$ is not necessarily zero. The solution of the initial problem is the sum of the solutions of the following problems.

*Problem* F. The plate surface is load-free: condition (2.1) is satisfied.

*Problem* G. There are no mass forces ($\mathbf{F} = 0$), and the distribution of the forces, according to condition (1.2), is specified on the plate surface.

For a plane uniform isotropic plate, we obtain the equations of flexural vibrations for Problems $F$ and $G$ in terms of the value of $V$, which we will call the FV and GV equations, and also in terms of the displacement of the middle plane $U = u_3(t, x_1, x_2, 0)$, which we will call the FU and GU equations.

We will consider the case when the forces applied to the surface of a plane of plate are directed along the normal, i.e.

$$A_{3j}\frac{\partial \mathbf{U}}{\partial x_j}\bigg|_{x_3 = \pm\varepsilon/2} = g_{\pm}(t, x_1, x_2)\mathbf{e}_3 \tag{4.1}$$

We will represent the solution in the form of the sum of solution with $\mathbf{g}_+^1 = -\mathbf{g}_-^1 = g\mathbf{e}_3/2$, where $g = g_+ - g_-$, and $\mathbf{g}_+^2 = -\mathbf{g}_-^2 = (g_+ + g_-)\mathbf{e}_3/2$. For plates possessing a definite symmetry about the plane $y_3 = 0$, in particular, plane uniform isotropic plates, in the second solution the displacement $u_3$ along the $x_3$ axis is odd about the $x_3 = 0$ plane, and hence the displacement of the middle plane of the plate along the $x_3$ axis is equal to zero: $u_3(t, x_1, x_2, 0) = 0$. The quantity $V$ – the average of $u_3$ over the cross-section, is also exactly equal to zero. Since we are investigating flexural vibrations, when constructing the GV and GU equations we will confine ourselves to considering the solution with the condition

$$\mathbf{g}_+ = -\mathbf{g}_- = g\mathbf{e}_3/2 \tag{4.2}$$

Construction of a special particular solution of the problem with non-homogeneous boundary conditions. To reduce the problem with non-homogeneous boundary conditions to a problem with homogeneous boundary conditions it is sufficient, for a specified vector function $\mathbf{g}$, to construct the vector function $\bar{\mathbf{U}}$, which satisfies the relations

$$L\bar{\mathbf{U}} = \mathbf{f}(t, x_1, x_2), \quad A_{ij}\frac{\partial \bar{\mathbf{U}}}{\partial x_j}\bigg|_{\partial\Pi} = \mathbf{g} \tag{4.3}$$

with a certain so-far unknown vector function $\mathbf{f}(t, x_1, x_2)$. In the case of a plane plate, only non-uniform in thickness, this problem has the following specific form: for specified $\mathbf{g}_\pm(t, x_1, x_2)$ we will seek a vector function $\bar{\mathbf{U}}(t, x_1, x_2, x_3/\varepsilon)$, which satisfies the relations

$$L\bar{\mathbf{U}} = -\rho\left(\frac{x_3}{\varepsilon}\right)D_t^2\bar{\mathbf{U}} + \frac{\partial}{\partial x_i}\left(A_{ij}\left(\frac{x_3}{\varepsilon}\right)\frac{\partial \bar{\mathbf{U}}}{\partial x_j}\right) = \mathbf{f}(t, x_1, x_2) \tag{4.4}$$

$$A_{3j}\frac{\partial \bar{\mathbf{U}}}{\partial x_j}\bigg|_{x_3 = \pm\varepsilon/2} = \mathbf{g}_\pm(t, x_1, x_2) \tag{4.5}$$

Equation (4.4) is equivalent to the equation

$$-\frac{\partial}{\partial x_3}\left(\rho\left(\frac{x_3}{\varepsilon}\right)D_t^2\right) + \frac{\partial^2}{\partial x_3 \partial x_i}\left(A_{ij}\left(\frac{x_3}{\varepsilon}\right)\frac{\partial \bar{\mathbf{U}}}{\partial x_j}\right) = 0 \tag{4.6}$$

We will replace the independent variable $x_3 = \varepsilon y$ in Eq. (4.6) and boundary conditions (4.5), and we will seek a solution in the form of the series

$$\bar{\mathbf{U}} \sim \sum_{n=1}^{\infty} \mathbf{U}^n(t, x_1, x_2, y)\varepsilon^n$$

Equating the coefficients of like powers of $\varepsilon$ in the equations obtained, we arrive at the system of equations

$$\begin{aligned}&\frac{\partial^2}{\partial y^2}\left(A_{33}\frac{\partial \mathbf{U}^n}{\partial y}\right) + \sum_{l=1,2}\left(\frac{\partial^2}{\partial y^2}\left(A_{3l}\frac{\partial \mathbf{U}^{n-1}}{\partial x_l}\right) + \frac{\partial^2}{\partial y \partial x_l}\left(A_{l3}\frac{\partial \mathbf{U}^{n-1}}{\partial y}\right)\right) + \\ &+ \sum_{k,l=1,2}\frac{\partial^2}{\partial y \partial x_k}\left(A_{kl}\frac{\partial \mathbf{U}^{n-2}}{\partial x_l}\right) - \frac{\partial}{\partial y}\left(\rho\frac{\partial^2 \mathbf{U}^{n-2}}{\partial t^2}\right) = 0\end{aligned} \tag{4.7}$$

and the boundary conditions

$$\left(A_{33}\frac{\partial \mathbf{U}^n}{\partial y} + A_{31}\frac{\partial \mathbf{U}^{n-1}}{\partial x_1} + A_{32}\frac{\partial \mathbf{U}^{n-1}}{\partial x_2}\right)\Big|_{y=\pm 1/2} = \delta_{n1}\mathbf{g}_{\pm}$$

Here $\mathbf{U}^{-1} = \mathbf{U}^0 = 0$. Equations (4.7) are ordinary differential equations in the variable $y$ for $\mathbf{U}^n$; the variables $t$ and $x$ occur in them as parameters. Consequently, beginning with $n = 1$, we can obtain the vector functions $\mathbf{U}^n(y)$ in quadratures. They are determined, apart from a constant term; we will always choose it so that the relation $\langle \mathbf{U}^n \rangle = 0$ is satisfied. After finding $\mathbf{U}^n$, using formula (4.4) we find $\mathbf{f} \sim L\bar{\mathbf{U}}$. A similar algorithm for constructing the required particular solution $\bar{\mathbf{U}}$ can be proposed in the general case (4.3); however, the function $\mathbf{U}^n$, as a rule, is then not necessarily found in quadratures. In the case of a plane uniform isotropic plate, with boundary conditions (4.2), the values of the components $\bar{U}_i$ and $f_i$ of the vector $\bar{\mathbf{U}}$ and $\mathbf{f}$ were calculated with a high degree of accuracy. In this case

$$\bar{U}_1|_{y=0} \equiv 0, \quad \bar{U}_2|_{y=0} \equiv 0, \quad f_1 \equiv 0, \quad f_2 \equiv 0$$

Below we present expressions for $f_3$ and $\bar{U}_3|_{y=0}$ with an error $O(\varepsilon^5)$

$$\begin{aligned} f_3 &= \frac{g}{\varepsilon} + \varepsilon\frac{\nu\Delta g}{12(1-\nu)} + \varepsilon^3\frac{(12\nu^2 - 10\nu - 1)D_t^2 - 2(\nu^2 - 10\nu - 1)\Delta}{1440(1-\nu)^2}\Delta g \\ \bar{U}_3|_{y=0} &= \varepsilon\frac{(2\nu - 1)g}{48(1-\nu)} + \varepsilon^3\frac{7(4\nu^2 - 4\nu + 1)D_t^2 + 4(13\nu^2 + 7\nu)\Delta}{23040(1-\nu)^2}g \end{aligned} \tag{4.8}$$

In the case of discontinuous $\rho$ and $A_{ij}$, instead of differential equations, we consider corresponding integral relations of the form (1.5).

*Reduction of the problem with non-homogeneous boundary conditions to a problem with homogeneous boundary conditions.* Suppose the FV equation is obtained and has the form

$$D_t^2 V \sim \hat{L}V + \hat{P}F_3 \tag{4.9}$$

with certain differential operators $\hat{L}$, $\hat{P}$ We will show how one can obtain from Eq. (4.9) equations corresponding to the other problems formulated above.

*The GV equation.* Suppose $\mathbf{u}$ is the solution of Problem G with $\mathbf{F} = 0$ and non-homogeneous boundary condition (4.1). The difference $\mathbf{w} = \mathbf{u} - \bar{\mathbf{U}}$ satisfies the equality $L\mathbf{w} \sim -\mathbf{f}$ c $\mathbf{f} \sim L\bar{\mathbf{U}}$ and the homogeneous boundary condition. Hence $V' \equiv \{w_3\}$ satisfies Eq. (4.9) with $F_3 = f_3$. Equalities (3.5) and (3.6) take the form

$$w_3 = u_3 - \bar{U}_3 \sim v_3' + Sv_3', \quad V' \equiv \{w_3\} \sim v_3' \tag{4.10}$$

Further

$$V \equiv \{u_3\} = \{\bar{U}_3 + w_3\} = V' \tag{4.11}$$

since $\{\bar{U}_3\} = 0$. Hence, we obtain the following equation for $V$

$$D_t^2 V \sim \hat{L}V + \hat{P}f_3 \tag{4.12}$$

*The FU equation.* By relations (3.5) and (3.6) we have

$$U \sim V + S_0 V, \quad S_0 V \sim \sum_{0<q+l}{}' \varepsilon^{2(q+l)} n_{2l}^{2q}(0) D_t^{2q} \Delta^l V \tag{4.13}$$

Applying the operator $E + S_0$ to Eq. (4.9) we obtain the equation

$$D_t^2 U \sim \hat{L}U + \hat{P}(E + S_0)F_3 \tag{4.14}$$

*The GU equation*. We will put $U_0(t, x_1, x_2) = \bar{U}_3(t, x_1, x_2, 0)$, and in the first of relations (4.10) $y_3 = 0$; taking the second of relations (4.10) and equality (4.11) into account, we obtain

$$U - U_0 \sim (E + S_0)V$$

We apply the operator $E + S_0$ to Eq. (4.12)

$$D_t^2(E + S_0)V \sim \hat{L}(E + S_0)V + (E + S_0)\hat{P}f_3$$

Adding this equality to the identity

$$D_t^2 U_0 = \hat{L}U_0 + (D_t^2 U_0 - \hat{L}U_0)$$

we obtain the equation

$$D_t^2 U \sim \hat{L}U + D_t^2 U_0 - \hat{L}U_0 + (E + S_0)\hat{P}f_3 \tag{4.15}$$

## 5. EQUATIONS OF ACCURACY $O(\varepsilon^3)$ AND $O(\varepsilon^4)$

Henceforth we will mean by equations of accuracy $O(\varepsilon^n)$ equations in which the neglected terms are of the order of $\varepsilon^n$. We then assume that **F** and **g** are of the order of unity. In fact, in different situations **F** and **g** may be small of different order. For example, in typical cases **g** is of the order of $\varepsilon$. Then, the accuracy of the equations derived below, containing **g**, will be one greater than indicated in the text.

Consider equations of accuracy $O(\varepsilon^3)$ and $O(\varepsilon^4)$. Equation (3.3), taking formulae (3.4) into account, can be written in the form

$$D_t^2 V = \varepsilon^2\left(\frac{1}{12}D_t^2\Delta - \frac{1}{6(1-\nu)}\Delta^2\right)V + F_3 + O(\varepsilon^4) \tag{5.1}$$

We will carry out equivalent transformations of Eq. (5.1) in order to eliminate the mixed derivative of $V$ with respect to $t$ and $x$. Differentiating Eq. (5.1), we obtain

$$D_t^2\Delta V = \Delta F_3 = O(\varepsilon^2)$$

Substituting this expression for $D_t^2\Delta V$ into Eq. (5.1), we obtain the following equation of the form FV of accuracy $O(\varepsilon^4)$

$$D_t^2 V \sim -\varepsilon^2\frac{1}{6(1-\nu)}\Delta^2 V + F_3 + \varepsilon^2\frac{1}{12}\Delta F_3 \tag{5.2}$$

From Eq. (5.2), proceeding in the same way as when deriving Eqs (4.12), (4.14) and (4.15), and using formulae (4.8), we obtain equations of the form GV of accuracy $O(\varepsilon^3)$

$$D_t^2 V = \varepsilon^2\frac{-1}{6(1-\nu)}\Delta^2 V + \frac{1}{\varepsilon}g + \varepsilon\frac{1}{12(1-\nu)}\Delta g$$

of the form FU of accuracy $O(\varepsilon^4)$

$$D_t^2 U = \varepsilon^2\frac{-1}{6(1-\nu)}\Delta^2 U + F_3 + \varepsilon^2\frac{2-3\nu}{24(1-\nu)}\Delta F_3$$

and of the form GU of accuracy $O(\varepsilon^3)$

$$D_t^2 U = \varepsilon^2\frac{-1}{6(1-\nu)}\Delta^2 U + \frac{1}{\varepsilon}g + \varepsilon\frac{(2\nu-1)D_t^2 + 2(2-\nu)\Delta}{48(1-\nu)}g \tag{5.3}$$

We recall that this equation is written in dimensionless variables. In dimensional variables it takes the form

$$\rho D_t^2 U = -\frac{H^2 E}{12(1-\nu^2)}\Delta^2 U + \frac{1}{H}g + H\frac{2(1+\nu)(2\nu-1)\rho D_t^2 + 2(2-\nu)E\Delta}{48(1-\nu)E}g$$

Here $E$ is Young's modulus. This equation differs from the classical Kirchhoff equation [2] in that it contains an additional term containing derivatives of $g$.

When $\mathbf{F} \neq 0$ and $\mathbf{g} \neq 0$ the required equations were obtained by adding the equations derived above.

## 6. THE EQUATIONS OF ACCURACY $O(\varepsilon^5)$ AND $O(\varepsilon^6)$

Before we obtain from Eq. (3.3) analogous equations of accuracy $O(\varepsilon^5)$ and $O(\varepsilon^6)$, we will consider the well-known equations of accuracy $O(\varepsilon^5)$.

*Selezov's equations.* For Problem G equations of the transverse vibrations of a plane uniform isotropic plate accuracy $O(\varepsilon^5)$ in terms of $U$ were obtained in [5] in the following form

$$\begin{aligned} D_t^2 U &= \varepsilon^2(a_0^4 D_t^4 + a_2^2 D_t^2 \Delta + a_4^0 \Delta^2)U + \\ &+ \varepsilon^4(a_0^6 D_t^6 + a_2^4 D_t^4 \Delta + a_4^2 D_t^2 \Delta^2 + a_6^0 \Delta^3)U + G \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} a_0^4 &= \frac{8\nu-7}{48(1-\nu)}, \quad a_2^2 = \frac{2-\nu}{6(1-\nu)}, \quad a_4^0 = -20a_6^0 = \frac{-1}{6(1-\nu)} \\ a_0^6 &= \frac{-64\nu^2+104\nu-41}{7680(1-\nu)^2}, \quad a_2^4 = \frac{16\nu^2-37\nu+19}{960(1-\nu)^2}, \quad a_4^2 = \frac{-4\nu^2+16\nu-11}{480(1-\nu)^2} \end{aligned} \tag{6.2}$$

and $G$ is a quantity, defined by the external forces applied to the plate surface.

$$\begin{aligned} G &= \frac{1}{\varepsilon}g + \varepsilon\frac{(1-\nu)D_t^2 + (\nu-2)\Delta}{8(1-\nu)}g + \\ &+ \varepsilon^3\frac{2(1-\nu)^2 D_t^4 - (4\nu^2-12\nu+7)D_t^2\Delta + 2(\nu^2-4\nu+3)\Delta^2}{768(1-\nu)^2}g \end{aligned} \tag{6.3}$$

We will show that Eq. (6.1), apart from terms $O(\varepsilon^5)$, can be converted to the following form, not containing differentiation of $U$ higher than the second order with respect to time and higher than the fourth order with respect to the coordinates

$$D_t^2 U = \hat{L}U + \hat{P}g \tag{6.4}$$

The operators $\hat{L}$ and $\hat{P}$ are defined by the formulae

$$\hat{L} = \varepsilon^2\left(-\frac{1}{6(1-\nu)}\Delta^2 + \frac{17-7\nu}{60(1-\nu)}D_t^2\Delta\right) \tag{6.5}$$

$$\begin{aligned} \hat{P} &= \frac{1}{\varepsilon} + \varepsilon\frac{5(2\nu-1)D_t^2 + 6(3\nu-8)\Delta}{240(1-\nu)} + \\ &+ \varepsilon^3\frac{7(2\nu-1)^2 D_t^4 + 2(28\nu^2+2\nu-29)D_t^2\Delta - 4(21\nu^2+64\nu-78)\Delta^2}{23040(1-\nu)^2} \end{aligned}$$

This form may turn out to be preferable, since in this case, for a correct formulation of the boundary-value problem, no additional initial and boundary conditions are required.

We will carry out the following transformations. We apply the operator $E + \varepsilon^2\left(a_0^4 D_t^2 - \frac{a_6^0}{a_4^0}\Delta\right)$ to Eq. (6.1) and add the result to Eq. (6.1); we obtain

$$D_t^2 U = \varepsilon^2\left(\left(a_2^2 + \frac{a_6^0}{a_4^0}\right)D_t^2\Delta + a_4^0\Delta^2\right)U + \varepsilon^4\left((a_0^6 + (a_0^4)^2)D_t^6 + \right.$$

$$+\left(a_2^4 + a_0^4 a_2^2 - \frac{a_6^0 a_4^0}{a_4^0}\right)D_t^4\Delta + \left.\left(a_4^2 + a_0^4 a_4^0 - \frac{a_6^0 a_4^0}{a_4^0}\right)D_t^2\Delta^2\right)U +$$

$$+ G + \varepsilon^2\left(a_0^4 D_t^2 - \frac{a_6^0}{a_4^0}\Delta\right)G + O(\varepsilon^5) \tag{6.6}$$

Further, applying the operators $\varepsilon^4 D_t^4$, $\varepsilon^4 D_t^2\Delta$, $\varepsilon^4\Delta^2$ to Eq. (6.1), we obtain the relations

$$\varepsilon^4 D_t^{6-2k}\Delta^k U = \varepsilon^4 D_t^{4-2k}\Delta^k G + O(\varepsilon^6), \quad k = 0, 1, 2$$

by means of which we can get rid of terms with derivatives of $U$ of the sixth order in Eq. (6.6). Finally, taking formulae (6.2) and (5.3) into account, we obtain Eq. (6.4).

*The equation of accuracy $O(\varepsilon^5)$ and $O(\varepsilon^6)$, obtained by the method of two-scale asymptotic expansions.* From Eq. (3.3) we have the equation

$$D_t^2 V = \varepsilon^2(A_2^2 D_t^2\Delta + A_4^0\Delta^2)V +$$
$$+ \varepsilon^4(A_2^4 D_t^4\Delta + A_4^2 D_t^2\Delta^2 + A_6^0\Delta^3)V + F_3 + O(\varepsilon^6)$$

which is identical in form with Eq. (6.1). Using the transformations employed to derive Eq. (6.4), and also employing expressions (3.4) for $A_l^q$, we obtain the following equation of the form FV of accuracy $O(\varepsilon^6)$ (everywhere in this section $\hat{L}$ is the operator defined by formula (6.5))

$$D_t^2 V = \hat{L}V + \hat{P}F_3 \tag{6.7}$$

where

$$\hat{P} = 1 + \varepsilon^2\frac{(\nu - 6)\Delta}{30(1-\nu)} - \varepsilon^4\frac{6(1-\nu)^2 D_t^2 + (\nu^2 + 12\nu - 12)\Delta}{720(1-\nu)^2}\Delta \tag{6.8}$$

Note that the difference $\delta$ of the right-hand sides of Eqs (6.7) and (5.2) is a quantity of the order of $\varepsilon^4$. In fact

$$\delta = \varepsilon^2\frac{17 - 7\nu}{60(1-\nu)}\Delta(D_t^2 V - F_3) + \varepsilon^4\frac{6(1-\nu)^2 D_t^2 - (\nu^2 + 12\nu - 12)\Delta}{720(1-\nu)^2}\Delta F_3$$

As a consequence of any of the equation (5.1) and (5.2) we have $D_t^2 V - F_3 = O(\varepsilon^2)$, and hence $\delta = O(\varepsilon^4)$.

After transformations similar to those employed in Section 4, we obtain an equation of the form GV of accuracy $O(\varepsilon^5)$

$$D_t^2 V = \hat{L}V + \hat{P}g$$

$$\hat{P} = \frac{1}{\varepsilon} + \varepsilon\frac{7\nu - 12}{60(1-\nu)}\Delta + \varepsilon^3\frac{(14\nu - 13)(D_t^2 - 2\Delta)}{1440(1-\nu)^2}\Delta$$

an equation of the form FU of accuracy $O(\varepsilon^6)$

$$D_t^2 U = \hat{L}U + \hat{P}F_3$$

$$\hat{P} = 1 - \varepsilon^2 \frac{\nu + 24}{120(1-\nu)}\Delta - \varepsilon^4 \frac{(150\nu^2 - 232\nu + 89)D_t^2 + 2(9\nu^2 + 88\nu - 89)\Delta}{11520(1-\nu)^2}\Delta$$

and an equation of the form GU of accuracy $O(\varepsilon^5)$, which is identical with Eq. (6.4), which, as was shown above, is equivalent to Selezov's equation with an accuracy up to terms $O(\varepsilon^5)$.

From relation (4.13) we obtain a relation of the form

$$V \sim \sum \varepsilon^{2(q+l)} b_{2l}^{2q} D_t^{2q} \Delta^l U$$

Then, substituting the vector $\mathbf{v} = (0, 0, V)$ into (3.5), we obtain expressions for the components of the vector $\mathbf{u}$ in terms of $U$

$$u_i \sim -\varepsilon y \frac{\partial U}{\partial x_i} + \varepsilon^3 y \frac{(4y^2 - 3)(\nu - 1)D_t^2 + 2(2(2-\nu)y^2 - 3)\Delta}{24(1-\nu)} \frac{\partial U}{\partial x_i}, \quad i = 1, 2$$

$$u_3 \sim U + \varepsilon^2 y^2 \frac{\nu}{2(1-\nu)}\Delta U +$$

$$+ \varepsilon^4 y^2 \frac{2y^2((1-2\nu^2)D_t^2 - 2(1-\nu^2)\Delta) + (6\nu^2 - 4\nu - 1)D_t^2 + 2(4\nu + 1)\Delta}{96(1-\nu)^2}\Delta U$$

These expansions, together with the well-known terms of order 1, $\varepsilon$ and $\varepsilon^2$ [1] contain terms of higher order in $\varepsilon$.

Note that, when deriving the equations of the vibrations of plates, considerably different equations are obtained, although of asymptotically equivalent form. For example, in the case of free vibrations, Eqs (3.3), (6.1) and (6.4) have the same accuracy – $O(\varepsilon^6)$. At the same time, it can be verified that, for the problem of the propagation of plane waves, four characteristics correspond to Eq. (3.3), six characteristics correspond to Eq. (6.1) and two characteristics correspond to Eq. (6.4). The choice of one form of equation or another depends on the specific problem and the purpose of the investigation.

## 7. THE EQUATIONS OF ACCURACY $O(\varepsilon^8)$

We will convert Eq. (3.3) to another form, in which derivatives of the fourth order with respect to time are retained, but derivatives of higher order are dropped, i.e. to the form of Timoshenko's equations [1]. In Eq. (3.3) we transfer $D_t^2 \upsilon_3$ to the right-hand side and apply the operators $\varepsilon^2\Delta$, $\varepsilon^2 D_t^2$, $\varepsilon^4\Delta^2$ and $\varepsilon^4 D_t^2\Delta$ to the equation obtained. Taking into account the fact that $\upsilon_3 \sim V$, dropping terms $O(\varepsilon^8)$, we have, respectively

$$0 = -\varepsilon^2 D_t^2 \Delta V + \varepsilon^4 (A_2^2 D_t^2 \Delta^2 + A_4^0 \Delta^3)V + \\ + \varepsilon^6 (A_2^4 D_t^4 \Delta^2 + A_4^2 D_t^2 \Delta^3 + A_6^0 \Delta^4)V + \varepsilon^2 \Delta F_3 \quad (7.1)$$

$$0 = -\varepsilon^2 D_t^4 V + \varepsilon^4 (A_2^2 D_t^6 \Delta + A_4^0 D_t^2 \Delta^2)V + \\ + \varepsilon^6 (A_2^4 D_t^6 \Delta + A_4^2 D_t^4 \Delta^2 + A_6^0 D_t^2 \Delta^3)V + \varepsilon^2 D_t^2 F_3 \quad (7.2)$$

$$0 = -\varepsilon^4 D_t^2 \Delta^2 V + \varepsilon^6 (A_2^2 D_t^2 \Delta^3 + A_4^0 \Delta^4)V + \varepsilon^4 \Delta^2 F_3 \quad (7.3)$$

$$0 = -\varepsilon^4 D_t^4 \Delta V + \varepsilon^6 (A_2^2 D_t^4 \Delta^2 + A_4^0 D_t^2 \Delta^3)V + \varepsilon^4 D_t^2 \Delta F_3 \quad (7.4)$$

We multiply relations (7.1), (7.2), (7.3) and (7.4) respectively by the unknown factors $V_2^0$, $V_0^2$, $V_4^0$ and $V_2^2$ and combine with relation (3.3); we obtain the equality

$$D_t^2 V = \varepsilon^2(Q_0^4 D_t^4 + Q_2^2 D_t^2 \Delta + Q_4^0 \Delta^2)V + \varepsilon^4(Q_2^4 D_t^4 \Delta + Q_4^2 D_t^2 \Delta^2 + Q_6^0 \Delta^3)V + \\ + \varepsilon^6(Q_2^6 D_t^6 \Delta + Q_4^4 D_t^6 \Delta^2 + Q_6^2 D_t^2 \Delta^3 + Q_8^0 \Delta^4)V + f_8 \quad (7.5)$$

where

$$\begin{aligned} &Q_0^4 = -V_0^2, \quad Q_2^2 = A_2^2 - V_2^0, \quad Q_4^0 = A_4^0, \quad Q_2^4 = A_2^4 + V_0^2 A_2^2 - V_2^2 \\ &Q_4^2 = A_4^2 + V_2^0 A_2^2 + V_0^2 A_4^0 - V_4^0, \quad Q_6^0 = A_6^0 + V_2^0 A_4^0 \\ &Q_2^6 = A_2^6 + V_0^2 A_2^4, \quad Q_4^4 = A_4^4 + V_2^0 A_2^4 + V_0^2 A_4^2 + V_2^2 A_2^2 \\ &Q_6^2 = A_6^2 + V_2^0 A_4^2 + V_0^2 A_6^0 + V_4^0 A_2^2 + V_2^2 A_4^0, \quad Q_8^0 = A_8^0 + V_4^0 A_4^0 + V_2^0 A_6^0 \\ &f_8 = F_3 + \varepsilon^2(V_0^2 D_t^2 + V_2^0 \Delta)F_3 + \varepsilon^4(V_2^2 D_t^2 \Delta + V_4^0 \Delta^2)F_3 \end{aligned} \quad (7.6)$$

It follows from Eq. (5.1) that

$$D_t^{2k+2}\Delta^l V = D_t^{2k}\Delta^l F_3 + O(\varepsilon^2) \ \forall k, l$$

Hence, Eq. (7.5) can be written, with accuracy $O(\varepsilon^8)$, in the form

$$D_t^2 V = \varepsilon^2(Q_2^2 D_t^2 \Delta + Q_0^4 D_t^4 + Q_4^0 \Delta^2)V + \\ + \varepsilon^4(Q_2^4 D_t^4 \Delta + Q_4^2 D_t^2 \Delta^2 + Q_6^0 \Delta^3)V + \varepsilon^6 Q_8^0 \Delta^4 V + \hat{P}F_3 \quad (7.7)$$

Here

$$\hat{P}F_3 = f_8 + \varepsilon^6(Q_2^6 D_t^4 \Delta + Q_4^4 D_t^2 \Delta^2 + Q_6^2 \Delta^3)F_3$$

*Timoshenko-type equations*. The equations of flexural vibrations in terms of the displacement of the middle plane of the plate, which has the form

$$D_t^2 U = \varepsilon^2(T_0^4 D_t^4 + T_2^2 D_t^2 \Delta + T_4^0 \Delta^2)U + G \quad (7.8)$$

where G is determined by the forces applied to the plate surface, are called Timoshenko-type equations. In order for Eq. (7.7) to have the form (7.8), the following equalities must be satisfied.

$$Q_2^4 = 0, \quad Q_4^2 = 0, \quad Q_6^0 = 0, \quad Q_8^0 = 0$$

It follows for these equations and relations (7.6) that

$$V_0^2 = \frac{422 - 424\nu - 33\nu^2}{4200(1-\nu)}, \quad V_2^0 = \frac{\nu - 6}{30(1-\nu)}, \quad V_2^2 = \frac{2 - 4\nu - 33\nu^2}{50400(1-\nu)}, \quad V_4^0 = -\frac{1}{12600} \quad (7.9)$$

After finding the quantities $V_j$ we obtain the non-zero quantities $Q_k^l$

$$Q_0^4 = \frac{33\nu^2 + 424\nu - 422}{4200(1-\nu)}, \quad Q_2^2 = \frac{17 - 7\nu}{60(1-\nu)}, \quad Q_4^0 = -\frac{1}{6(1-\nu)}, \quad Q_2^6 = \frac{33\nu^2 - \nu + 3}{504000(1-\nu)} \\ Q_4^4 = \frac{-66\nu^4 + 268\nu^3 - 247\nu^2 + 64\nu - 24}{1008000(1-\nu)^3}, \quad Q_6^2 = \frac{67\nu^2 - 24\nu + 12}{504000(1-\nu)^2} \quad (7.10)$$

Changing from the FV equation to the GU equation, we have

$$D_t^2 U = \hat{L} U + \hat{P} g \tag{7.11}$$

where

$$\hat{L} = \varepsilon^2\left(-\frac{1}{6(1-\nu)}\Delta^2 + \frac{17-7\nu}{60(1-\nu)}D_t^2\Delta + \frac{33\nu^2+424\nu-422}{4200(1-\nu)}D_t^4\right)$$

$$\hat{P} = \frac{1}{\varepsilon} - \frac{\varepsilon}{2800(1-\nu)}((22\nu^2+166\nu-223)D_t^2 + 70(3\nu-8)\Delta) -$$

$$-\frac{\varepsilon^3}{268800(1-\nu)^2}((88\nu^3+760\nu^2-1364\nu-481)D_t^4 + 2(44\nu^3+804\nu^2-$$

$$-1670\nu+787)D_t^2\Delta - 4(157\nu^2-384\nu+227)\Delta^2 - 4(157\nu^2-384\nu+227)\Delta^2) -$$

$$-\frac{\varepsilon^5}{64512000(1-\nu)^3}((616\nu^4+5232\nu^3-12538\nu^2+8306\nu-1711)D_t^6 -$$

$$-2(616\nu^4+5736\nu^2-15864\nu^2+12345\nu-2866)D_t^4\Delta +$$

$$+(616\nu^4-56\nu^3-12817\nu^2+15548\nu-2976)D_t^2\Delta + 8(737\nu^3-856\nu^2+231\nu-162)\Delta^3)g$$

Another version of the simple form of Eqs (7.7) arises if it is required to satisfy the equations

$$Q_0^4 = 0,\quad Q_2^4 = 0,\quad Q_6^0 = 0,\quad Q_8^0 = 0$$

Then $V_0^2 = 0$ and $V_2^2 = -1/20$, while the values of $V_2^0$ and $V_4^0$ as before, are defined by (7.9), $Q_2^2$ and $Q_4^2$ are the same as in formulae (7.10), and

$$\begin{aligned} Q_4^2 &= \frac{422-424\nu-33\nu^2}{25200(1-\nu)^2}, \quad Q_2^6 = \frac{17}{20160} \\ Q_4^4 &= \frac{216\nu^3-1758\nu^2+2768\nu-1231}{302400(1-\nu)^3}, \quad Q_6^2 = \frac{721+1027\nu-268\nu^2-3\nu^3}{151200(1-\nu)^3} \end{aligned} \tag{7.12}$$

In the equations thereby obtained there is no differentiation of $U$ (or $V$) higher than the second order with respect to time and, consequently, when formulating the boundary-value problem, there is no need for additional initial conditions. However, as shown in Section 9, the Cauchy problem for these equations is incorrect.

## 8. THE CHOICE OF THE PARAMETERS IN TIMOSHENKO'S EQUATIONS

The values of $T_l^q$ and $G$ in Timoshenko-type equations (7.8) were derived in [1]; if we change to the dimensionless variables (1.3) and (3.2) with scales (3.1), the values of $T_l^q$ and $G$ are obtained as follows:

$$\begin{aligned} T_0^4 &= \frac{-1}{12K_1}, \quad T_2^2 = \frac{1}{12} + \frac{1}{6(1-\nu)K_2}, \quad T_4^0 = Q_4^0 = -\frac{1}{6(1-\nu)} \\ G &= \frac{1}{\varepsilon}g + \varepsilon\left(\frac{D_t^2}{12K_3} - \frac{\Delta}{6(1-\nu)K_4}\right)g \end{aligned} \tag{8.1}$$

where $K_1 = K_2 = K_3 = K_4 = K$ is a certain coefficient; its value in Timoshenko's model is discussed, for example, in [4, 5]. In Reissner's model for the statics case $K = 5/6$.

We will write the values $K_j$ in relations (8.1), for which, in the case of free oscillations ($\mathbf{F} = 0$, $\mathbf{g} = 0$), Eq. (7.8) with coefficients (8.1) is identical with Eqs (7.11) with accuracy $O(\varepsilon^8)$, while in the case of oscillations under the action of forces applied to the plate surfaces ($g \neq 0$), it has an accuracy $O(\varepsilon^3)$. We will put $K_j = 5\omega_j(\nu)/6$ and require that the equalities $T_0^4 = Q_0^4$, $T_2^2 = Q_2^2$ must be satisfied and also that there should be equality between the terms of order $\varepsilon^{-1}$, $\varepsilon$ and $\varepsilon^3$ in Eqs (7.11) and (7.8) with coefficients defined by formulae (8.1). We then obtain

$$\omega_1(\nu) = \frac{420(1-\nu)}{422-424\nu+33\nu^2}, \quad \omega_2(\nu) = \frac{6}{6-\nu}$$

$$\omega_3(\nu) = \frac{280(1-\nu)}{223-166\nu-22\nu^2}, \quad \omega_4(\nu) = \frac{8}{8-3\nu}$$

The functions $\omega_j(\nu)$ are monotonic in the section [0, 0.5]; their values at the ends of the interval are

$$\omega_1(0) \approx 0.99890, \quad \omega_1(0.5) \approx 0.99533; \quad \omega_2(0) = 1, \quad \omega_2(0.5) \approx 1.09091$$

$$\omega_3(0) \approx 1.04089, \quad \omega_3(0.5) \approx 1.25561; \quad \omega_4(0) = 1, \quad \omega_4(0.5) \approx 1.06667$$

Hence, the assumption that $K = 5/6$ does not necessitate any very considerable correction, namely, when introducing the coefficients $K_j = 5\omega_j(\nu)/6$ to obtain the equations of free oscillations of accuracy $O(\varepsilon^8)$ and the equations of forced oscillations due to the action of forces applied to the plate surface, of accuracy $O(\varepsilon^3)$. At the same time, Eqs (7.8) with coefficients (8.1) when $K = 5/6$ have an accuracy $O(\varepsilon^2)$ in the case of free oscillations, and accuracy $O(\varepsilon)$ in the case of forced oscillations.

## 9. INVESTIGATION OF THE CORRECTNESS OF THE EQUATIONS OF FLEXURAL VIBRATIONS

To investigate the correctness of the equations of flexural vibrations (5.3) of accuracy $O(\varepsilon^4)$, (6.4) of accuracy $O(\varepsilon^6)$ and (7.7) for $Q_{2l}^{2k}$, defined by (7.12), of accuracy $O(\varepsilon^8)$, and (7.11) of accuracy $O(\varepsilon^8)$, it is sufficient to investigate the case of free vibrations ($\mathbf{F} = 0$, $\mathbf{g} = 0$). Consider the first three of these equations

$$\begin{aligned} &1)\ D_t^2 U = \varepsilon^2 Q_4^0 \Delta^2 U, \quad 2)\ D_t^2 U = \varepsilon^2 Q_2^2 D_t^2 \Delta U + \varepsilon^2 Q_4^0 \Delta^2 U, \\ &3)\ D_t^2 U = \varepsilon^2 Q_2^2 D_t^2 \Delta U + \varepsilon^2 Q_4^0 \Delta^2 U + \varepsilon^4 Q_4^2 D_t^2 \Delta^2 U \end{aligned} \tag{9.1}$$

where $Q_4^0$, $Q_2^2$, $Q_4^2$ are defined by (7.10) and (7.12). Note that when $0 \le \nu \le 0.5$ the following inequalities are satisfied

$$Q_4^0 < 0, \quad Q_2^2 > 0, \quad Q_4^2 > 0 \tag{9.2}$$

Since the derivatives with respect to the variable $x_1$ and $x_2$ occur in these equations only as part of the Laplace operator, to investigate the problem of the correctness of the Cauchy problem it is sufficient to consider particular solutions of the form $U = e^{\sigma t - ikx_1}$. Substituting these particular solutions into the corresponding equations (9.1), we obtain the equalities

$$\begin{aligned} &1)\ \sigma^2 = Q_4^0 \varepsilon^2 k^4, \quad 2)\ \sigma^2(1 + Q_2^2(\varepsilon k)^2) = Q_4^0 \varepsilon^2 k^4, \\ &3)\ \sigma^2(1 + Q_2^2(\varepsilon k)^2 - Q_4^2(\varepsilon k)^4) = Q_4^0 \varepsilon^2 k^4 \end{aligned}$$

It follows from (9.2) that when $k \ge 0$ is the first two case $\sigma^2 \le 0$ and, consequently, $\sigma$ is a pure imaginary quantity, i.e. the correctness condition is satisfied. In the third case $\sigma^2 \to \infty$, if $\varepsilon k$ approaches the root of the equation $1 + Q_2^2(\varepsilon k)^2 - Q_4^2(\varepsilon k)^4 = 0$ and, consequently, the Cauchy problem is ill-posed.

Consider Eq. (7.11) in the case of free vibrations. It has the form

$$D_t^2 U = \varepsilon^2 (Q_0^4 D_t^4 + Q_2^2 D_t^2 \Delta + Q_4^0 \Delta^2) U$$

The coefficients $Q_0^0$, $Q_2^2$ and $Q_4^0$ are defined by (7.10). When $0 \le \nu \le 0.5$ we have

$$Q_0^4 < 0, \quad Q_2^2 > 0, \quad Q_4^0 < 0$$

The equation $Q_0^4\sigma^4 - (1 + Q_2^2(\varepsilon k)^2)\sigma^2 + Q_4^0\varepsilon^2 k^4 = 0$ in $\sigma$ for all real $k$ has four pure imaginary roots, and hence the corresponding Cauchy problem satisfies the necessary condition for correctness.

## 10. CONVERSION OF THE EQUATIONS TO A FORM WHERE THE RIGHT-HAND SIDE IS INDEPENDENT OF FAST DISPLACEMENTS

The procedure for constructing Eqs (2.7) for the problem of the vibrations of periodic plates, described in Section 2, is directly applicable when the right-hand side of $\mathbf{F}$ is independent of the fast variables $y$. With the exception of the case of a uniform plate, the right-hand side, as a rule, depends on the fast variables, for example, $\mathbf{F} = \rho(y_1, y_2, y_3)g\mathbf{e}$, where $g$ is the acceleration due to gravity and $\mathbf{e}$ is a certain vector.

To reduce the problem with the vector function $\mathbf{F} = \mathbf{F}(t, x_1, x_2, y_1, y_2, y_3))|_{y_j = x_j/\varepsilon}$, smooth with respect to the variables $t, x_1, x_2$ and 1-periodic with respect to the variables $y_1, y_2$, to a problem with the vector function $\mathbf{F} = \hat{\mathbf{F}}(t, x_1, x_2)$, smooth with respect to the variables $t, x_1, x_2$, we construct the vector function $\bar{\mathbf{U}}$ such that

$$L\bar{\mathbf{U}} \sim \hat{\mathbf{F}}(t, x_1, x_2) - \mathbf{F}(t, x_1, x_2, y_1, y_2, y_3)|_{y_j = x_j/\varepsilon} \tag{10.1}$$

To do this we simultaneously seek the asymptotic expansions

$$\hat{\mathbf{F}} \sim \sum_{n \ge 0} \mathbf{f}^n(t, x_1, x_2)|_{y_j = x_j/\varepsilon}\varepsilon^n, \quad \bar{\mathbf{U}} \sim \sum_{n \ge 0} \mathbf{U}^n(t, x_1, x_2, y_1, y_2, y_3)|_{y_j = x_j/\varepsilon}\varepsilon^n \tag{10.2}$$

with $\mathbf{U}^0 = \mathbf{U}^1 = 0$. We substitutes series (10.2) into relations (10.1). Equating coefficients of like powers of $\varepsilon$, we obtain

$$L_{yy}\mathbf{U}^2 = \mathbf{f}^0 - \mathbf{F}, \quad L_{yy}\mathbf{U}^{n+2} + L_1\mathbf{U}^{n+1} + L_0\mathbf{U}^n = \mathbf{f}^n \text{ when } n > 0$$
$$L_1 = \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial}{\partial y_j}\right) + \frac{\partial}{\partial y_i}\left(A_{ij}\frac{\partial}{\partial x_j}\right), \quad L_0 = \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial}{\partial x_j}\right) - \rho D_t^2 \tag{10.3}$$

The operator $L_{yy}$ is defined by formula (2.3)

For each $n$ it is sufficient to obtain at least one pair $\mathbf{U}^{n+2}$, $\mathbf{f}^n$ which satisfies relations (10.3). In many cases this chain of equations is solvable if $\mathbf{f}^0 = \langle \mathbf{F} \rangle$ and $\mathbf{f}^n = \langle L_1\mathbf{U}^{n+1} + L_0\mathbf{U}^n \rangle$ when $n > 0$. The difference $\mathbf{w} = \mathbf{u} - \bar{\mathbf{U}}$ will be the solution of the equation

$$L\mathbf{w} = -\hat{\mathbf{F}}(t, x_1, x_2)$$

## 11. SUBSTANTIATION OF THE EQUATIONS OBTAINED

Using the example of Eq. (6.7) we will indicate the main features of the rigorous substantiation of the correctness of the equations obtained.

Consider the boundary-value problem for the system of equations (1.1) in a plane uniform plate with uniform boundary conditions (2.1) and initial conditions

$$\mathbf{u}|_{t=0} = 0, \quad \mathbf{u}_t|_{t=0} = 0$$

We will denote by $\Omega(T)$ the set of points $(t, x_1, x_2, x_3)$, lying in the zone $0 \le t \le T$, and by $\Omega_0(T)$ the set of points $(t, x_1, x_2, 0)$, lying in the zone $0 \le t \le T$. We will further assume that the vector function $\mathbf{F} = \mathbf{F}(t, x_1, x_2)$ satisfies the following conditions:

(1) $F_1 = F_2 \equiv 0$;

(2) the third component $F_3$ of the vector $\mathbf{F}$ is finite and, after continuing it to zero, when $t < 0$, it belongs to $H^8_{\text{loc}}(R_4)$.

It can be shown by the method of energy estimates that for the function $V$, which is the solution of Eq. (6.7), the following limit holds

$$\|V\|_{H^8(\Omega_0(T))} \le c_1(T)\|F_3\|_{H^8(\Omega_0(T))} \tag{11.1}$$

Here and henceforth $c_i(T)$ are certain quantities which depend only on $T$.

Equation (6.7) can be regarded as having been obtained from the initial equation (3.3) by using the operation $\hat{P}$ (6.8) and dropping terms of higher order. It can be verified that, after applying the operation $(\hat{P})^{-1}$ to both sides of Eq. (6.7) we obtain a certain equation

$$\partial^2 V/\partial t^2 = L'V + F_3'$$

with the following properties. Terms of the order of $\varepsilon^k$ ($k = 0, \dots, 4$) in $L'$ are identical with the corresponding terms in Eq. (3.3), and $F_3' = F_3 + O(\varepsilon^6)$. We put

$$\bar{\mathbf{V}} = (0, 0, V)^T, \quad \bar{\mathbf{F}} = (0, 0, f_3)^T$$

$$\bar{\mathbf{u}} \sim \sum_{q+l_1+l_2 \le 6}' \varepsilon^{q+l_1+l_2} N^q_{l_1 l_2}(y)\big|_{y = x/\varepsilon} \frac{\partial^{q+l_1+l_2}\bar{\mathbf{V}}}{\partial t^q \partial x_1^{l_1} \partial x_2^{l_2}}$$

Since, in the case of a plane uniform plate, system of equations (2.7) can be split into a system of equations in $\upsilon_1$ and $\upsilon_2$ and an equation in $\upsilon_3$, the following inequality holds

$$\|L(\mathbf{u} - \bar{\mathbf{u}})\|_{H^0(\Omega(T))} \le c_2(T)\varepsilon^{13/2}(\|V\|_{H^8(\Omega_0(T))} + \|F_3\|_{H^6(\Omega_0(T))})$$

Taking the limit (11.1) into account, we rewrite this inequality in the form

$$\|L(\mathbf{u} - \bar{\mathbf{u}})\|_{H^0(\Omega(T))} \le c_3(T)\varepsilon^{13/2}\|F_3\|_{H^8(\Omega_0(T))}$$

Further, by the standard method of energy estimates we obtain that

$$\|\mathbf{u} - \bar{\mathbf{u}}\|_{H^0(\Omega(T))} \le c_4(T)\varepsilon^{13/2}\|F_3\|_{H^8(\Omega_0(T))}$$

Hence we obtain the following estimate of the closeness of $u_3$ and $V$

$$\|\{u_3\} - V\|_{H_0(\Omega_0(T))} = O(\varepsilon^6)c_5(T)\varepsilon^6\|F_3\|_{H^8(\Omega_0(T))}$$

Condition 2, imposed on the function $F_3$, can be replaced by a weaker one as a result of carrying out a more detailed estimate.

The above constructions show that the a priori assumption that an asymptotic expansion of the solution **u** in a series of the given form in powers of $\varepsilon$ is justified under certain conditions.

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